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Some recurrence relations and their parallel evaluation using nested recurrent product form algorithm

Ayşe KIPER
Department of Computer Engineering, Middle East Technical University, Ankara, Turkey

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Abstract: Recurrence relations for the Fourier expansion of the Jacobian elliptic functions $\mathrm{sn}^m(u, k)$, $\mathrm{cn}^m(u, k)$, $\mathrm{dn}^m(u, k)$ with $m \geqslant 1$ and those for the expansion of $\sin^m(\pi x)$, $\cos^m(\pi x)$ in powers of the Jacobian elliptic functions with $m \geqslant 1$ are studied and their parallel evaluation is formulated using nested recurrent product form algorithm.

Keywords: Jacobian elliptic functions, ellipsoidal wave equation, Lamé equation, unit time step, recursive doubling.

1. Introduction

In the evaluation of functions using series approximations, the accuracy increases with the increasing number of terms involved in the expansion which results in an increase in computer time. In the last decade, the advances in the area of "Parallel Computing" have led the numerical analysts to develop parallel numerical algorithms for the solution of various problems, including the recurrence relations.

The Fourier series expansion for the twelve Jacobian elliptic functions (JEFs) have been studied and given by several authors [1–3,10]. The recurrence formulae have been obtained for the coefficients of these series corresponding to powers of the JEFs [5]. Among the applications which involve the Fourier expansion of $\mathrm{sn}^2(x, k)$ and $\mathrm{sn}^4(x, k)$, the characteristic solutions of the ellipsoidal wave equation [9] and those of the Lamé equation [7] may be mentioned. The diffraction problem of scalar waves by elliptic discs and apertures is based on these solutions [4]. The similarity between the behaviour of $\sin(x)$ and $\mathrm{sn}(x, k)$ and those of $\cos(x)$ and $\mathrm{cn}(x, k)$ leads to considering the expansions for $\sin^m(\pi x)$ and $\cos^m(\pi x)$ with $m \geqslant 1$ in powers of the JEFs $\mathrm{sn}(2Kx, k)$ and $\mathrm{cn}(2Kx, k)$ respectively.

In this paper the recurrence relations for the expansion coefficients of:
(i) the Fourier series for powers of the JEFs,
(ii) the series for powers of the trigonometric functions $\sin(\pi x)$ and $\cos(\pi x)$ in JEFs,
are considered and their parallel evaluation is formulated in the form of a nested recurrent product form algorithm.

2. Some expansion coefficients and their generalization

Recurrence formulae for the expansion coefficients of the Fourier series for the JEFs $\mathrm{sn}^m(u, k)$, $\mathrm{cn}^m(u, k)$ and $\mathrm{dn}^m(u, k)$, with $m = 1, 2, 3, \ldots$, and those for the expansion coefficients of the trigonometric functions $\sin^m(\pi x)$ and $\cos^m(\pi x)$, with $m = 1, 2, 3, \ldots$, have been obtained [5,6]. Properties of the recurrence relations allow us to develop a general form for each group and then to study their parallel evaluation.

2.1. Recurrence formulae for the Fourier series coefficients for powers of the JEFs

The analysis of the relations for the Fourier series coefficients for powers of the Jacobian elliptic functions (JEFs) [5] shows that for a prescribed k (k is the modulus of the elliptic functions) they may be represented by the common expression,

$$\left.\begin{aligned} &\Omega_n^{(0)} = \alpha, \\ &\Omega_n^{(1)} = \beta, \\ &\Omega_n^{(r)} = a(n, r)\Omega_n^{(r-1)} + b(r)\Omega_n^{(r-2)}, \quad r = 2, 3, \ldots, l, \end{aligned}\right\} \quad n = 0, 1, 2, \ldots, m, \tag{1}$$

where l is the required power and m is the required number of terms.

The relation given by (1) can be expressed as the solution of a matrix system

$$\begin{bmatrix} \Omega_n^{(2)} \\ \Omega_n^{(3)} \\ \Omega_n^{(4)} \\ \vdots \\ \Omega_n^{(l)} \end{bmatrix} = \begin{bmatrix} 0 & & & & & 0 \\ a(n,3) & 0 & & & & \\ b(4) & a(n,4) & 0 & & & \\ 0 & & & & & \\ \vdots & & & & & \\ 0 & & & b(l) & a(n,l) & 0 \end{bmatrix} \begin{bmatrix} \Omega_n^{(2)} \\ \Omega_n^{(3)} \\ \Omega_n^{(4)} \\ \vdots \\ \Omega_n^{(l)} \end{bmatrix} + \begin{bmatrix} a(n,0)\Omega_n^{(1)} + b(0)\Omega_n^{(0)} \\ b(1)\Omega_n^{(0)} \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{2}$$

If the first m terms of an expansion for a power l are required, then the matrix system of size $(l-1)$ must be solved $(m+1)$ times with $n = 0, 1, 2, \ldots, m$. The size of the system increases with the increasing required power, and the number of solutions of the system increases with the increasing number of terms. It must be noted that the system allows us to determine the coefficients for a prescribed power, in any order, since the systems for $n = 0, 1, 2, \ldots, m$ are independent.

2.2. Recurrence formulae for the expansion of $\sin^m(\pi x)$ *and* $\cos^m(\pi x)$ *with* $m = 1, 2, \ldots$ *in series of JEFs*

The expansion of $\sin^m(\pi x)$ in powers of $\mathrm{sn}(2Kx, k)$ and that of $\cos^m(\pi x)$ in powers of $\mathrm{cn}(2Kx, k)$ have been studied and the related recurrence relations for the expansion coefficients have been given [6]. These recurrences can be generalized as

$$w_{n+2}^{(m)} = a(m, n)w_n^{(m)} + b(n)w_{n-2}^{(m)} + c(n, m)w_n^{(m-2)}, \quad n = m, m+2, \ldots, m+2p, \tag{3}$$

with the starting value $w_m^{(m)} = (w_0)^m$, and $m = 1, 2, \ldots, l$, where l is the required power, and p is the required number of terms.

The relation (3) can also be expressed as the solution of a matrix system

$$\begin{bmatrix} w_{m+2}^{(m)} \\ w_{m+4}^{(m)} \\ w_{m+6}^{(m)} \\ w_{m+8}^{(m)} \\ \vdots \\ w_{m+2p}^{(m)} \end{bmatrix} = \begin{bmatrix} 0 & & & & & 0 \\ a(m, m+2) & 0 & & & & \vdots \\ b(m+4) & a(m, m+4) & 0 & & & \\ 0 & b(m+6) & a(m, m+6) & 0 & & \\ \vdots & & & & & \\ 0 & & b(m+2(p-1)) & a(m, m+2(p-1)) & 0 & \end{bmatrix} \begin{bmatrix} w_{m+2}^{(m)} \\ w_{m+4}^{(m)} \\ w_{m+6}^{(m)} \\ w_{m+8}^{(m)} \\ \vdots \\ w_{m+2p}^{(m)} \end{bmatrix}$$
$$+ \begin{bmatrix} a(m, m)w_m^{(m)} + c(m, m)w_m^{(n-2)} \\ b(m+2)w_m^{(m)} + c(m, m+2)w_{m+2}^{(m-2)} \\ c(m, m+4)w_{m+4}^{(m-2)} \\ c(m, m+6)w_{m+6}^{(m-2)} \\ \vdots \\ c(m, m+2(p-1))w_{m+2(p-1)}^{(m-2)} \end{bmatrix}. \tag{4}$$

If p is the required number of coefficients for a specified even power l (odd power l), the system with p unknowns must be solved in the order $m = 4, 6, 8, \ldots, l$ ($m = 3, 5, 7, \ldots, l$) starting with the coefficients of $m = 2$ ($m = 1$). Therefore the solution of a system depends on the previous system solution, resulting in a nested solution of systems of equations.

3. Parallel determination of the expansion coefficients

Matrix systems (2) and (4) resulting from the recursions studied in Section 2 have the general form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_l \end{bmatrix} = \begin{bmatrix} 0 & & & & 0 \\ a_{21} & 0 & & & \\ a_{31} & a_{32} & 0 & & \\ 0 & & & & \\ \vdots & & & & \\ 0 & 0 & a_{l,l-2} & a_{l,l-1} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_l \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_l \end{bmatrix}, \tag{5}$$

which is a second-order, linear recurrence system ($\mathrm{R}\langle l, 2\rangle$) having small order compared to the size of the system. One of the fastest procedures for the solution of the system is the "recurrent product form algorithm" [8, p.44].

The system (5), $\boldsymbol{x} = A\boldsymbol{x} + \boldsymbol{c}$, may be represented by the product form

$$\boldsymbol{x} = (I - A)^{-1}\boldsymbol{c} = L^{-1}\boldsymbol{c}, \tag{6}$$

with

$$L^{-1} = M_l M_{l-1} \cdots M_1, \tag{7}$$

where

$$M_i = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & -a_{i+1,i} & & \\ & -a_{i+2,i} & & \\ & 0 & \ddots & \\ & \vdots & & \\ 0 & 0 & & 1 \end{bmatrix}. \tag{8}$$

In (7), $M_l = I$, then the solution takes the product form

$$\boldsymbol{x} = M_{l-1} M_{l-2} \cdots M_1 \boldsymbol{c}. \tag{9}$$

Solution vector $\boldsymbol{x}$ can be calculated in parallel by use of a recursive doubling procedure.

In accordance with the related theorem [8, p.44], the necessary unit time steps T_p and the number of processors P for the evaluation of the system have been obtained by an exact calculation

$$T_p \leqslant 3 \log_2 l - 1, \qquad P \leqslant 3l - 8. \tag{10}$$

The independency of the systems in (2) gives rise to the parallel solution of these systems and each system is solved by the recurrent product form algorithm in parallel using a recursive doubling procedure. Therefore the unit time steps are proportional only to the power but the necessary number of processors are proportional not only to the power but also to the number of terms.

Because of the dependency of the solutions of system (5), the nested form of the recurrent product form algorithm can be applied. As it can be seen in (10), the nested solution does not increase the number of processors but increases the unit time steps proportionally to the power.

The accuracy of the numerical results depends heavily on that of K, E and q (K and E are the complete elliptic integrals of the first and the second kind respectively and the quantity $q = \exp(-\pi K'/K)$ is referred to as the nome). Therefore these quantities must be computed as accurately as possible prior to doing the computations of these coefficients themselves.

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